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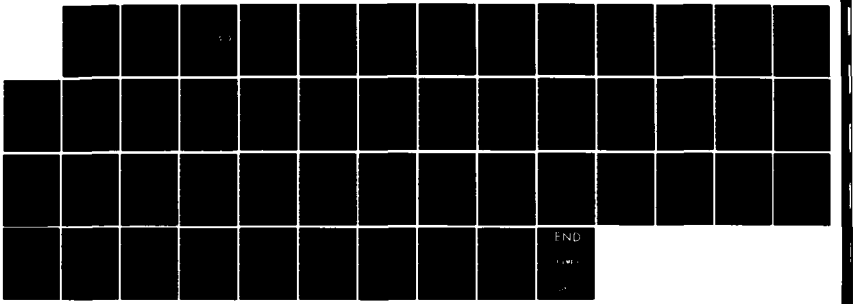

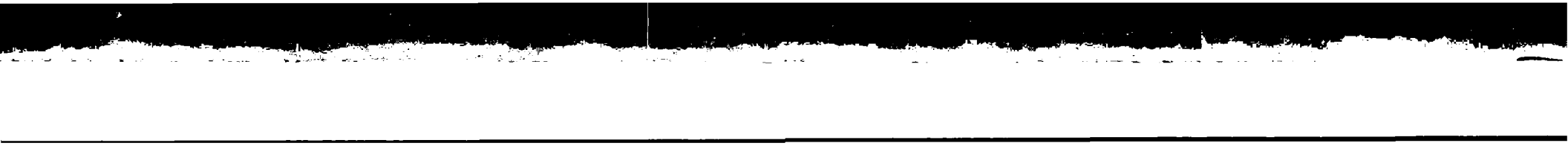

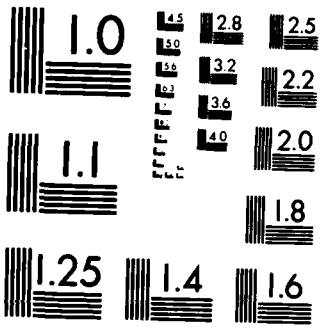

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for Estimating Network Reliability

George S. Fishman

Technical Report No. UNC/ORSA/TR-84/8
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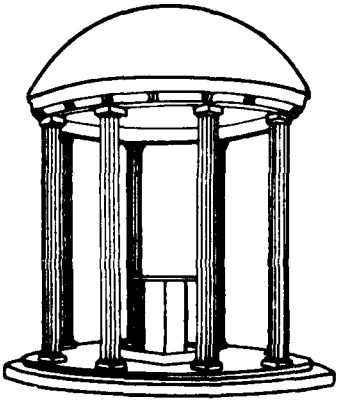
AT CHAPEL HILL



A Monte Carlo Sampling Plan for Estimating Network Reliability

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This research was supported in part by the Air Force Office of Scientific Research under Contract AFOSR-84-0140. Reproduction in whole or in part is permitted for any purpose of the United States government.

Abstract

Consider an acyclic undirected network G = (V,E) with node set V and arc set E whose arcs are subject to random failure. Let s be a node in V and T a set of nodes in V such that s ∉ T. This paper presents a relatively complete and comprehensive description of a general class of Monte Carlo sampling plans for estimating g = g(s,T), the probability that s is connected to all nodes in T. The paper also provides procedures for implementing these plans. Each plan uses known lower and upper bounds [B,A] on g to produce an estimator of g that has a smaller variance (A-g)(g-B)/K than one obtains for crude Monte Carlo sampling (B=0, A=1) on K independent replications. The paper describes worst case bounds on sample sizes K, in terms of B and A, for meeting absolute and relative error criteria. It also gives the worst case bound on the amount of variance reduction that can be expected when compared with crude Monte Carlo sampling.

Two plans are studied in detail for the case T = {t}. An example illustrates the variance reductions achievable with these plans. The paper next shows how to assess the credibility that a specified error criterion for g is met as the Monte Carlo experiment progresses and then shows how confidence intervals can be computed for g. Lastly, the paper summarizes the steps needed to implement the proposed technique.

Introduction

Consider an acyclic undirected network $G = (V,E)$ with node set V and arc set E. Suppose that nodes are perfect but that arcs are subject to random failure. Let s be a node in V and T a set of nodes in V such that $s \notin T$. The purpose of this paper is to present a relatively complete and comprehensive description of a general class of Monte Carlo sampling plans for estimating $g(s,T)$, the probability that s is connected to all nodes in T. The paper also provides procedures for implementing these plans. In the single node case with $T = \{t\}$, we speak of s-t connectedness; when $T = V - \{s\}$, we speak of network connectedness.

Since it is well known that direct computation of $g(s,T)$ generally requires exponential time, a variety of alternative methods have been proposed to approximate $g(s,T)$. Notable among these proposals are those based on bounds and on Monte Carlo methods. As an example of bounds, Esary and Proschan (1963, Thm. 4.1) describe a lower bound that derives from a network $G_1 = (V,E)$ in which all minimal cutsets are disjoint and describe an upper bound that derives from a network $G_2 = (V,E)$ in which all minimal paths are disjoint.

Any attempt to use bounds raises two important questions. First, are the bounds sufficiently tight to forgo the need for a more precise computation of $g(s,T)$ and second, how much time does the computation of the bounds require? Note that if the computation of bounds for a proposed method takes exponential time in either $|V|$ or $|E|$ in general, then the question of tightness is

merely an academic one. This is so for the Esary and Proschan bounds.

Monte Carlo sampling is the other major approach to approximating g(s,T). Whereas all techniques of evaluation accumulate numerical error as the size of the network grows, the Monte Carlo method additionally introduces sampling error. To appreciate the significance of this error, we note that, if each arc is either on or off, there are $2^{|E|}$ possible states of the network. Therefore, K replications of a Monte Carlo experiment can at best account for only a small number of these states when $|E|$ is even moderate in size, and this limitation introduces sampling error.

Since sampling error for independent trials or replications decreases as $1/K^{1/2}$, it is clear that more replications means a smaller error. Therefore, the cost of a Monte Carlo depends on K which in turn depends on the level of sampling error one is prepared to accept and on the cost per replication. Two concepts of sampling error arise in practice, one absolute and the other relative. Let $\bar{g}_K$ denote an unbiased estimate of $g = g(s,T)$ based on K replications. Then to achieve an absolute error no larger than $\varepsilon > 0$ with confidence level greater than $1-\delta$, one needs to collect

$$K = \min[k: \ pr(|\bar{g}_k - g| > \varepsilon) \leq \delta] \qquad (1)$$

replications. For a relative error no greater than ε on g one needs

$$K = \min[k: \ pr\left(\frac{|\hat{g}_k - g|}{g} > \varepsilon\right) \leq \delta] \qquad (2)$$

replications. For a relative error no greater than ε on the failure probability $1 - g$ one needs

$$K = \min[k: \ pr\left(\frac{|\hat{g}_k - g|}{1-g} > \varepsilon\right) \leq \delta] \qquad (3)$$

replications.

A major consideration in using the Monte Carlo method is to achieve one or more of these error objectives while keeping K and the cost per replication within reason. Variance reducing techniques described in Van Slyke and Frank (1972), Kumamoto, Tanaka and Inoue (1977), Easton and Wong (1980), Kumamoto, Tanaka, Inoue and Henley (1980), Karp and Luby (1983) and Fishman (1983a, 1983b) are all intended to make K smaller than it otherwise would be if crude Monte Carlo sampling were used. Among these only the Karp and Luby proposal addresses the issue of how cost grows as a function of δ and ε. In particular, their method achieves (3) in $O(|E| m)$ time where m denotes the number of failure sets of the network. Since this approach requires the determination and storage of all m failure sets prior to performing the sampling experiment and since m can be large in practice, the approach has a clear limitation. A shortcoming of all other approaches is that none provides an <u>a priori</u> estimate of how large K needs to be in order to achieve a specified error bound.

The present paper introduces the general concept of a Monte Carlo sampling plan that relies explicitly on the use of a priori bounds $0 < B \leq g(s,T) \leq A < 1$ to estimate $g(s,T)$ unbiasedly. Section 1 describes the plan and shows that every such sampling plan leads to a smaller variance var $\bar{g}_K$ and to smaller coefficients of variation $\gamma(\bar{g}_K; A,B) = (\text{var } \bar{g}_K)^{1/2}/g$ and $\gamma(1-\bar{g}_K; A,B) = (\text{var } \bar{g}_K)^{1/2}/(1-g)$ than obtain for crude Monte Carlo sampling. Moreover, worst case bounds can be computed for all these quantities in terms of A and B. These worst case bounds enable one to compute upper bounds on K for criteria (1), (2) and (3) thus providing valuable information prior to beginning the sampling experiment. The paper next derives the worst case lower bound on the variance reduction achievable when compared with crude Monte Carlo sampling.

Each sampling plan has three major time related cost considerations: 1) the time to compute A and B prior to sampling, 2) the time to sample the status of each arc in E on each replication and 3) the time to determine connectedness on each replication. With regard to this last cost, we assume that a depth-first search algorithm as in Aho, Hopcroft and Ullman (1974, p. 176) is used which takes $O(\max(|V|, |E|))$ time.

Section 2 shows that these general sampling plans include the Van Slyke and Frank (1972) bounds as a special case. When failures occur independently with identical failure probabilities, the computation of A and B takes $O(|E|)$ time and sampling

per replication also takes $O(|E|)$ time. Section 3 shows how the method of Kumamoto, Tanaka and Inoue (1977) also fits into this class of sampling plans. Since their suggested implementation can require exponential time to compute A and B and more than $O(|E|)$ time per replication for sampling, we describe an alternative approach that takes $O(|E|)$ time for computing A and B and for sampling per replication. For fixed δ and ε the results for bounds and timings lead to (1) in $O(\max(|V|, |E|)(A-B)^2)$ time, to (2) in $O(\max(|V|, |E|)(A-B)^2/AB)$ time and to (3) in $O(\max(|V|, |E|)(A-B)^2/(1-A)(1-B))$ time.

Section 4 illustrates how the proposed sampling plan works for estimating s-t connectedness for a 30 arc network for the Van Slyke and Frank (VSF) bounds and for the Kumamoto, Tanaka and Inoue (KTI) bounds. Section 5 then shows how to assess the credibility that a specified error criterion for 1-g is met as K increases. Section 6 next shows how one can derive confidence intervals for g by using exact sampling theory, Chebyschev-like bounds and a normal approximation. Section 7 summarizes all the steps needed to implement a Monte Carlo sampling plan using the KTI bounds.

We begin with several useful definitions. For each $i \varepsilon E$ let

$x_i = 1$ if arc i operates

$= 0$ otherwise

$x = (x_i, i \varepsilon E)$ = a state of the network

X_j = set of all states x such that

$$\sum_{i \in E} x_i \leq j \qquad j = 0,1,\ldots,|E|$$

p_i = probability that arc i operates

and

$P(x)$ = probability that state x occurs.

Observe that if failures occur independently then

$$P(x) = \prod_{i \in E} [1 - p_i + x_i(2p_i - 1)] \qquad x \in X = X^{|E|}. \quad (4)$$

Lastly, we define the structure function

$\phi(x) = \phi(x; s,T) = 1$ if node s is connected to all nodes in T

$= 0$ otherwise

and the s-T connectedness probability

$$g = g(s,T) = \sum_{x \in X} \phi(x)\, P(x).$$

Note that if $T = \{t\}$ then the existence of an operating path from s to t implies connectedness. If $T = V - \{s\}$, then the existence of a spanning tree implies connectedness.

1. Sampling Plans

Let Γ denote the set of all sampling plans $\{\psi(x), Q(x);\ x \in X\}$ where each $\{\psi(x)\}$ is a binary function of the form

$$\psi(x) = (A-B)\phi(x) + B \qquad 0 < B \leq g \leq A < 1 \quad (5)$$

and $\{Q(x)\}$ is a sampling distribution such that

$$\sum_{x \in X} \psi(x)Q(x) = g,$$

or equivalently,

$$\sum_{x \in X} \phi(x)Q(x) = (g-B)/(A-B).$$

If one draws K independent samples $x^{(1)},\ldots,x^{(K)}$ from $\{Q(x)\}$ then

$$\bar{g}_K = \frac{1}{K} \sum_{j=1}^{K} \psi(x^{(j)}) \tag{6}$$

is an unbiased estimator of g with

$$\text{var } \bar{g}_K = (A-g)(g-B)/K \leq (A-B)^2/4K \tag{7a}$$

and coefficients of variation

$$\gamma(\bar{g}_K;\ A,B) = \frac{(\text{var } \bar{g}_K)^{1/2}}{g} \leq (A-B)/2(KAB)^{1/2} \tag{7b}$$

and

$$\gamma(1-\bar{g}_K;\ A,B) = \frac{(\text{var } \bar{g}_K)^{1/2}}{1-g} \leq (A-B)/2[K(1-A)(1-B)]^{1/2}. \tag{7c}$$

These results which follow from maximization of $(A-g)(g-B)$, $(A-g)(g-B)/g^2$ and $(A-g)(g-B)/(1-g)^2$, respectively, have several notable features. Observe that they apply for dependent as well as independent arc failures. Also, observe that $\min(\,|A-g|\,,\,|g-B|\,)$ moreso than A-B determines the magnitude of var $\bar{g}_K$, revealing the greater benefit of a single tight bound as compared to a small interval A-B. For an absolute error ε as in (1) one has

$$K \leq [\beta(\delta)(A-B)/2\varepsilon]^2 \tag{8}$$

where

$$\beta(\delta) = \min\{\beta: \ pr[\frac{|\bar{g}_K - g|}{(\text{var } \bar{g}_K)^{1/2}} > \beta] \leq \delta\}.$$

For a relative error ε in (2) and (3) one has, respectively,

$$K \leq [\beta(\delta)(A-B)/2\varepsilon]^2/AB \tag{9}$$

and

$$K \leq [\beta(\delta)(A-B)/2\varepsilon]^2/(1-A)(1-B). \tag{10}$$

To compute the bounds in (8), (9), and (10) one needs to know $\{\beta(\delta), \ 0 < \delta < 1\}$. Although $S = K(\bar{g}_K - B)/(A-B)$ has the binomial distribution with parameters K and $\mu = (g-B)/(A-B)$, this is of limited value at this point since μ is unknown. From Chebyshev's inequality one has

$$\beta(\delta) = 1/\delta^{1/2}, \tag{11}$$

which leads to a conservatively larger upper bound than is generally required.

Observe that as $K \to \infty$ the distribution of $(S-K\mu)/[K\mu(1-\mu)]^{1/2}$ converges to the standard normal distribution. Since K needs to be large when ε is small, one may in such a case use

$$\beta(\delta) = [\beta: \ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\beta} e^{-z^2/2} dz = (1-\delta)/2] \tag{12}$$

in (8), (9) and (10), but noting its approximating nature. These bounds on K provide a convenient prior assessment of worst case effort and make clear the desirability of striving for short intervals [B,A].

Procedure BOUNDS describes the steps needed to perform K independent replications on a Monte Carlo experiment using a sam-

pling plan in Γ. Observe that $\bar{g}_K$ is the maximum likelihood estimator of g and $V(\bar{g}_K)$, apart from the division by K - 1 instead of K, is the maximum likelihood estimator of var $\bar{g}_K$. The use of K - 1 in place of K makes $V(\bar{g}_K)$ unbiased in small samples.

Procedure BOUNDS

Purpose: To estimate g(s,T).

Input: Network G = (V,E,s,T), bounds A and B, sampling distribution $\{Q(x),\ x\in X\}$ and sample size K.

Output: Point estimate $\tilde{g}_K$ of g(s,T) and point estimate $V(\tilde{g}_K)$ of var $\tilde{g}_K$.

Method:

I. Initialization

Start with $S \equiv 0$.

II. On each replication 1,...,K do:

a. Determine state by sampling x from $\{Q(x)\}$.

b. Check for s-T connectedness; if connected add 1 to S.

III. Computation of final estimates

a. $\bar{g}_K = (A-B)\ S/K + B.$

b. $V(\tilde{g}_K) = (A-B)^2(1-S/K)(S/K)/(K-1).$

End of procedure.

1.1 Crude Monte Carlo Sampling

The significance of a sampling plan $\{\psi(x),Q(x)\}$ can stood best by first considering the simplest form of Mont sampling. Here $A = 1$, $B = 0$ and $Q(x) = P(x)$ so that $\psi(x^{(1)}),\ldots,\psi(x^{(K)})$ are the outcomes of K independent Ber trials with parameter g and

$$\text{var } \tilde{g}_K = g(1-g)/K.$$

If arc failures are independent then P(x) has the form (4 that determining the status of each arc is itself an inde Bernoulli trial.

Now observe that $g(1-g)/(A-g)(g-B)$ gives the number cations needed to achieve the same variance with crude Mo Carlo sampling as one obtains using the bounds B and A. fore the worst relative performance, with regard to varia a sampling plan based on B and A occurs when this quantit minimal; namely at

$$g' = 1/\{1+[(1-A)(1-B)/AB]^{1/2}\}$$

so that $g'(1-g')/(A-g')(g'-B)$ provides an <u>a priori</u> lower the potential variance reduction.

For the remainder of the paper we assume arc failure independent. Procedure B describes how one samples from $P(x)$, using $W = E$ as input in $O(|E|)$ time.

Procedure B

Purpose: To sample the status of each arc $i \in W$.

Input: W and $\{p_i, i \in W\}$.

Output: $x = \{x_i, i \in W\}$.

Method:

I. For each $i \in W$ do: sample U_i from $U(0,1)$ and set

$$x_i = \lfloor p_i + U_i \rfloor .$$

End of procedure.

2. Bounds Based on Minimal Cardinality Paths and Cutsets

Crude Monte Carlo sampling makes no use of prior information regarding the network under study, a weakness that can be removed, at least conceptually, with minimal effort. For the network G, there exists a set of arcs $M = M(s,T)$ such that at least $|M|$ arcs must operate in order for s-T connectedness to be possible. Also, there exists a set of arcs $C = C(s,T)$ such that at least $|C|$ arcs must fail in order for s-T disconnectedness to be possible. For example, for $T = \{t\}$ M denotes the s-t path of minimal cardinality and C denotes the s-t cutset of minimal cardinality. For $T = V - \{s\}$, M denotes the s-T spanning tree of minimal cardinality and C denotes the network cutset of minimal cardinality.

The significance of M and C is that for every integer $L \leq |M|$

$$\phi(x) = 0 \qquad \text{for } x \in X_{L-1}$$

and for every integer $H \leq |C|$

$$\phi(x) = 1 \qquad \text{for } x \in X_{|E|} - X_{|E|-H}.$$

To incorporate this information into the sampling procedure, one sets

$$A = 1 - \sum_{x \in X_{L-1}} P(x) \tag{13a}$$

$$B = 1 - \sum_{x \in X_{|E|-H}} P(x) \tag{13b}$$

and samples x using the distribution

$$Q(x) = P(x)/(A-B) \qquad x \in X_{|E|-H} - X_{L-1}. \tag{14}$$

To benefit from these bounds, one first needs to know $|M|$ and $|C|$. The determination of M takes $O(|E|)$ time and of C takes $O(|V|^{2/3}|E|)$ time (Papadimitriou and Steiglitz, 1982, Th. 9.3, p. 213). Presumably L and H are then chosen so that the calculation of A and B are computationally feasible. For example, if $p_i = p$ $i \in E$ then with ease one can choose $L = |M|$ and $H = |C|$ and compute

$$A = 1 - F_{L-1}(|E|, p)$$

and

$$B = 1 - F_{|E|-H}(|E|, p),$$

where

$$F_i(n,\theta) = \sum_{j=0}^{i} \binom{n}{j} \theta^j (1-\theta)^{n-j} \qquad 0 < \theta < 1 \qquad 0 \le i \le n,$$

in $O(|E|)$ time. By contrast, for unequal p_i one needs to perform $O(\sum_{i=0}^{L} \binom{|E|}{i})$ steps to compute A and $O(\sum_{i=0}^{|E|-H+1} \binom{|E|}{i})$ steps to compute B, tasks that may be burdensome if $|M|$ and $|C|$ are large and one takes $L = |M|$ and $H = |C|$. When this is so, choosing smaller L and H reduces the burden, but also widens the interval A-B.

In contrast to the independent Bernoulli sampling for each arc's status in Procedure B, the present procedure calls for sampling x on each replication subject to the constraint

$$L \leq \sum_{i \in E} x_i \leq |E| - H. \qquad (15)$$

For the case of $p_i = p$ $i \in E$, Procedure BIN can effect this sampling, with $n = |E|$, $\theta = p$, $a = |H|$, $b = |L|$ and $W = E$ as input, in $O(|E|)$ time. In particular, step IIb determines the number of operating arcs in O(1) time if one uses the cutpoint method of sampling from a discrete distribution as described in Fishman and Moore (1984). This method requires the preparation of two tables prior to doing any sampling. These tables, whose computation take $O(|E|)$ time, are constructed in such a way that subsequent sampling on each replication takes constant time, independent of $|C|$, $|E|$ and $|M|$. As an alternative one may sample k in step IIb using the alias method of Walker (1977) as described in Kronmal and Peterson (1979) in the same time.

Once k is known, there are $\binom{|E|}{k}$ ways of assigning the k oper-

ating arcs among the $|E|$ arcs. Step IIIb selects the combination m that is to be used on this replication in O(1) time and step IV uses the k-canonical representation of the integer m (Kruskal 1963 and Katona 1966) to identify the arcs that operate in $O(|E|)$ time. Additional details about this sampling method appear in Fishman (1983b).

Van Slyke and Frank (1972) first proposed using the bounds in (13) to reduce variance in Monte Carlo experiments in the case of independent arc failures with $p_i = p$ $i \in E$. Hereafter we call these the VSF bounds. An extension of this approach is also possible. One takes $L > |M|$, $H > |C|$ and sets

$$A = 1 - \sum_{x \in X_{|M|-1}} P(x) - \sum_{x \in X_L - X_{|M|}} \phi(x)\, P(x) \qquad (16a)$$

and

$$B = 1 - \sum_{x \in X_{|E-C|}} P(x) - \sum_{x \in X_{|E-C|} - X_{|E|-H}} \phi(x) P(x). \qquad (16b)$$

The idea here is that one selects L and H and uses the resulting A,B and $\{Q(x)\}$, as constrained by (15), to effect a greater variance reduction than $L = |M|$ and $H = |C|$ allow. This nibbling away at the reliability computation has an increasingly beneficial effect as $\min_i p_i \to 1$ since the term for which $\sum_{i \in E} x_i = |E-C|$ becomes important in (16a). It has a similar effect when $\max_i p_i \to 0$ since the term with $\sum_{i \in E} x_i = |M|$ becomes important in (16b). The feasibility of using this method depends on the ability to compute

Procedure BIN

Purpose: To sample the status of each arc i in W.

Input: n,θ, $\{F_i(n,\theta);\ i = 0,1,\ldots,n\}$, $a,b,W = \{w_1,\ldots,w_n\}$.

Output: $x = \{x_i,\ i\in W\}$.

Method:

I. Initialization

a. For each $i\in W$, set $x_i = 0$.

II. Determine the number of arcs that operate

a. Sample U from U(0,1).

b. $k = \min\{j: F_j(n,\theta) - F_{b-1}(n,\theta) > U[F_{n-a}(n,\theta) - F_{b-1}(n,\theta)];\ b \le j \le a\}$.

III. Determine the "number" of the combination

a. Sample U from U(0,1).

b. $m = \lfloor \binom{n}{k} U \rfloor$.

IV. Determine status of arcs

a. $\omega = k$ and $\Delta = m$.

b. Until $\Delta = 0$ do: $y = \max[z: \binom{z}{\omega} \le \Delta]$;
$i = w_{y+1}$; $x_i = 1$; $\Delta = \Delta - \binom{y}{\omega}$; and $\omega = \omega - 1$.

c. Until $\omega = 0$ do: $i = w_\omega$; $x_i = 1$; and $\omega = \omega - 1$.

End of procedure.

$\phi(x)$ for each $x \in X_L \neg X_{|M|}$ and $x \in X_{|E-C|} \neg X_{|E|} \neg H$. Since the computational feasibility of A and B in (16) are in question even for the special case of equal p_i, the general applicability of this approach is even more limited.

3. Bounds Based on a Control

The bounds [B,A] in Section 2 are global in nature and exploit a relatively small amount of information about the network under study. Moreover, their application is severely limited by the fact that only in the case of equal probabilities are the bounds B and A easily calculated and the arc states easily sampled. The present section considers more informative bounds that rely on the status of a subset of arcs and shows how the resulting local conditionality enables one again to achieve a variance reduction. This approach is originally due to Kumamoto, Tanaka and Inoue (1977), but the presentation here is considerably different in organization and more comprehensive in character. We refer to the resulting bounds as the KTI bounds.

Let $\{\phi_1(x)\}$ and $\{\phi_2(x)\}$ denote binary functions on $\{0,1\}$ such that

$$\phi_1(x) \leq \phi(x) \leq \phi_2(x). \tag{17}$$

Then for

$$g_i = \sum_{x \in X} \phi_i(x)\, P(x) \qquad i = 1,2$$

one has

$$g_1 \leq g \leq g_2.$$

Observe that

$$\phi_1(x)\phi(x) = 1 \quad \text{if} \quad \phi_1(x) = 1$$
$$= 0 \quad \text{if} \quad \phi_1(x) = 0$$

and

$$\phi_2(x)\phi(x) = 1 \quad \text{if} \quad \phi(x) = 1$$
$$= 0 \quad \text{if} \quad \phi(x) = 0$$

so that

$$\phi_1(x)\phi(x) = \phi_1(x)$$

and

$$\phi_2(x)\phi(x) = \phi(x).$$

Let

$$Q(x) = \frac{\phi_2(x) - \phi_1(x)}{g_2 - g_1} P(x) \qquad x \in X \qquad (18)$$

and suppose one samples x from $\{Q(x)\}$. Then

$$\sum_{x \in X} \phi(x) Q(x) = \frac{g - g_1}{g_2 - g_1},$$

which suggests that one take $B = g_1$ and $A = g_2$. Then var $\psi(x)$ $= (g_2 - g)(g - g_1) < g(1 - g)$.

Two remaining issues concern the selection of $\{\phi_1(x)\}$ and $\{\phi_2(x)\}$ and the method of sampling from $\{Q(x)\}$ in (18). For

$T = \{t\}$, Kumamoto, Tanaka and Inoue suggest that one can determine a lower bound $\{\phi_1(x)\}$ based on minimal s-t paths and an upper bound $\{\phi_2(x)\}$ based on minimal s-t cutsets. In particular, let $P_1,\ldots,P_I$ denote I minimal s-t paths and $C_1,\ldots,C_J$, J minimal s-t cutsets for the network $G = (V,E)$. Then (17) holds for a coherent system if one defines the bounds as

$$\phi_1(x) = 1 - \prod_{j=1}^{I} (1 - \prod_{i \in P_j} x_i) \tag{19a}$$

and

$$\phi_2(x) = 1 - \prod_{j=1}^{J} [1 - \prod_{i \in C_j} (1-x_i)]. \tag{19b}$$

Observe that (19a) and (19b) act as <u>controls</u> on the values that the structure function $\{\phi(x)\}$ can assume.

To implement this approach, one needs to determine $P_1,\ldots,P_I$ and $C_1,\ldots,C_J$, compute g_1 and g_2 and devise a sampling plan for x. Kumamoto, Tanaka and Inoue address only the sampling issue. Let

$$\Omega = (\bigcup_{j=1}^{I} P_j) \cup (\bigcup_{j=1}^{J} C_j)$$

so that sampling x_i for $i \in E-\Omega$ involves a Bernoulli trial. Sampling arcs in Ω takes more care. Kumamoto, Tanaka and Inoue suggest sampling these arcs sequentially. Define two disjoint arc sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 + \Omega_2$ and define

$$\phi_1(\Omega_1,\Omega_2) = 1 - \prod_{j=1}^{I} (1 - \prod_{i \in P_j \cup \Omega_1} x_i \prod_{i \in P_j \cup \Omega_2} p_i)$$

and

$$\phi_2(\Omega_1,\Omega_2) = \prod_{j=1}^{J}[1-\prod_{i\in C_j\cup\Omega_1}(1-x_i)\prod_{i\in C_j\cup\Omega_2}(1-p_i)].$$

Then

$$pr(x_r = 1) = \frac{[\phi_2(\emptyset,\Omega_2-\{r\})-\phi_1(\emptyset,\Omega_2-\{r\})]p_r}{g_2 - g_1} \qquad r\in\Omega_2$$

and (20)

$$pr(x_r = 1 \mid x_i,\ i\in\Omega_1) = \frac{[\phi_2(\Omega_1,\Omega_2-\{r\})-\phi_1(\Omega_1,\Omega_2-\{r\})]p_r}{\phi_2(\Omega_1,\Omega_2) - \phi_1(\Omega_1,\Omega_2)}.$$

$$|\Omega_1| > 0,\ r\in\Omega_2.$$

One sees that the evaluation of $\phi_2(\cdot,\cdot)$ and $\phi(_1(\cdot,\cdot)$ can take $0(\sum_{j=1}^{J}|C_j|)$ time and $0(\sum_{j=1}^{I}|P_j|)$ time, respectively, so that for all arcs in Ω these computations take $0(|\Omega|(\sum_{j=1}^{J}|C_j| + \sum_{j=1}^{I}|P_j|))$ time.

Without any further specification, the computation of g_1 and g_2 takes $0(2^{\sum_{j=1}^{I}|P_j|})$ and $0(2^{\sum_{j=1}^{J}|C_j|})$ time, respectively, and, in fact, the time required to determine I paths and J cutsets remains in question.

To reduce sampling time when the determination of $P_1,\ldots,P_I$, $C_1,\ldots,C_J$, g_1 and g_2 are computationally feasible we suggest the following more efficient method. Observe that

$$Q(x) = Q(y,z) = Q_1(y)Q_2(z)$$

where

$$y = \{x_i, \ i\epsilon\Omega\}, \ z = \{x_i, \ i\epsilon E\neg\Omega\},$$

$$Q_1(y) = \frac{\phi_2(y)-\phi_1(y)}{g_2-g_1} \prod_{i\epsilon\Omega} [x_i(2p_i-1)+1-p_i]$$

and

$$Q_2(z) = \prod_{i\epsilon E-\Omega} [x_i(2p_i-1)+1-p_i].$$

Now one can compute the

$$n = \sum_{y\epsilon Y} [\phi_2(y) - \phi_1(y)] \qquad (21)$$

entries for $\{Q_1(y)\}$ in $O(2^{|\Omega|})$ time. Provided this is computationally feasible, one can then use these entries to compute tables for the cutpoint method of Fishman and Moore (1984), also in $O(2^{|E|})$ time, prior to experimentation. Lastly, one needs to create a 1-1 mapping from the n outcomes to their corresponding stored vectors $y = (x_i, i\epsilon\Omega)$. Then on each replication sampling from this $\{Q_1(y)\}$ occurs in $O(|\Omega|)$ time.

As the size of the network grows the effectiveness of this method of variance reduction can only be maintained if Ω grows with E. Therefore, there comes a point at which the tabling method is no longer feasible; nor are the determination of $P_1,\ldots,P_I$, $C_1,\ldots,C_J$, g_1 and g_2 computationally feasible without additional restrictions. To solve this problem we propose that the paths $P_1,\ldots,P_I$ be

chosen edge-disjoint and the cutsets $C_1,\ldots,C_J$ be chosen edge-disjoint. This offers immediate advantages.

First, it is clear that $I \leq |C|$, C being the s-t cutset of minimal cardinality. In fact, one can achieve $I = |C|$ and determine $P_1,\ldots,P_I$ in $O(I \cdot |E|)$ time using a network flow algorithm with unit capacities, as described in Wagner (1975, p. 954). Second, $J \leq |M|$ where M denotes the s-t path of minimal cardinality. Here one can determine the $J = |M|$ disjoint cutsets in $O(|E|)$ time by beginning at node s and appropriately labeling arcs. Third, edge-disjointness implies that g_1 and g_2 are computable in $O(\sum_{j=1}^{I} |P_j|)$ and $O(\sum_{j=1}^{J} |C_j|)$ times respectively.

Fourth, one can sample all arcs in E in $O(|E|)$ time. Procedure Q describes such an algorithm. In particular, it uses the quantities:

$$\lambda_0 = 0$$

$$\lambda_j = \prod_{i \in P_j} p_i \qquad j = 1,\ldots,I$$

$$\omega_0 = 0$$

$$\omega_j = \prod_{i \in C_j} (1-p_i) \qquad j = 1,\ldots,J$$

$$\lambda = 1 - \prod_{j=1}^{I} (1-\lambda_j)$$

$$\omega = \prod_{j=1}^{J} (1-\omega_j)$$

$$j_i = j \qquad \text{if } i \in P_j \qquad j = 1,\ldots,I$$

$$= 0 \qquad \text{if } i \notin \sum_{k=1}^{I} P_k$$

and

$$k_i = j \qquad \text{if } i \in C_j \qquad j = 1,\ldots,J.$$

$$= 0 \qquad \text{if } i \notin \sum_{k=1}^{J} C_k.$$

Here $\{j_i\}$ and $\{k_i\}$ are pointers identifying the path and cut-set, respectively, to which arc i belongs.

4. Bounds on Bounds

If one uses a table or Procedure Q then an (ε,δ) absolute criterion has $O(\max(|V|, |E|)[\beta(\delta)(A-B)/2\varepsilon]^2)$ worst case time from (7a) and (ε,δ) relative criteria have $O(\max(|V|, |E|)[\beta(\delta)(A-B)/2\varepsilon]^2/AB)$ and $O(\max(|V|, |E|)[\beta(\delta)(A-B)/2\varepsilon]^2/(1-A)(1-B))$ worst case times from (7b) and 7c) respectively. While (19) leads to a variance reduction, it is important to understand exactly how these worst case times behave as a function of $\{p_i, i\in\Omega\}$ and as a function of the size of the network G. For convenience of exposition we assume $|P_1| \leq \ldots \leq |P_I|$ and

Procedure Q

Purpose: To sample the status of each arc i in E.

Input: $E, \{p_i, i \in E\}$, I, J, $\{\lambda_j; j=0,1,\ldots,I\}$, $\{\omega_j; j=0,1,\ldots,J\}$, λ, ω, $\{j_i, k_i; i \in \Omega\}$.

Output: $\{x_i, i \in E\}$.

Method:

I. Initialization: $\Delta = \omega$ and $\rho = \lambda$.

II. Sample arcs in $E-\Omega$

If $p_i = p$ for all $i \in E-\Omega$, then use Procedure BIN to determine status of these arcs.

Otherwise: For each arc i in $E-\Omega$, sample U from $U(0,1)$ and set $x_i = \lfloor p_i + U \rfloor$.

III. Sample arcs in Ω

For each arc i in Ω do:

a. Compute the probability that arc i fails

$$\omega = \frac{1}{1-\omega_{k_i}} [1-\omega_{k_i}/(1-p_i)]; \quad \lambda = (\lambda-\lambda_{j_i})/(1-\lambda_{j_i})$$

and $q = (\omega-\lambda)p_i/(\Delta-\rho)$.

b. Determine the state of arc i

Sample U from U(0,1) and set $x_i = \lfloor U-q+1 \rfloor$.

c. Update parameters for the next arc

$$\omega = \omega/[1-x_i\omega_{k_i}/(1-p_i)]; \quad \omega_{k_i} = (1-x_i)\omega_{k_i}/(1-p_i);$$

$$\lambda = 1-(1-\lambda)[1-x_i\lambda_{j_i}/p_i]; \quad \lambda_{j_i} = \lambda_{j_i} x_i/p_i;$$

$\Delta = \omega$ and $\rho = \lambda$.

End of procedure.

$|C_1| \leq \ldots \leq |C_J|$, note that $|P_1| \geq |M|$, $I \leq |C|$, $|C_1| \geq |C|$ an $|M|$, and begin with $\gamma(\bar{g}_K; A,B)$ in (7b).

4.1 Boundedness with Respect to $\{p_i, i \in \Omega\}$

Observe that for disjoint paths

$$B = g_1 = 1 - \prod_{j=1}^{I} (1 - \prod_{i \in P_j} p_i) \qquad (23$$

and let

$$r = \max_{1 \leq j \leq I} \prod_{i \in P_j} p_i$$

and

$$N = \text{number of paths } k \text{ for which } \prod_{i \in P_k} p_i = r \quad k = 1,..$$

Then one can write (23) in the more concise form

$$B = Nr + o(r) \qquad (2$$

where $o(w)$ denotes a function $h(w)$ such that $\lim_{w \to 0} \frac{h(w)}{w} = 0$.

Now observe that for disjoint cutsets

$$A = g_2 = \prod_{j=1}^{J} [1 - \prod_{i \in C_j} (1-p_i)] \qquad (2$$

and let

$$q_j = \max_{i \in C_j} p_i \qquad j=1,\ldots,J$$

$$q = \prod_{j=1}^{J} q_j$$

and

$$M_j = \text{number of } p_i = q_j \quad i \in C_j \quad j = 1,\ldots,J.$$

Then A has the more concise representation

$$A = q \prod_{j=1}^{J} M_j + o(q). \tag{26}$$

Note that (25) and (26) hold regardless of whether or not $P_1,\ldots,P_J$ are disjoint and $C_1,\ldots,C_J$ are disjoint.

With regard to (7b) one has for a given network G

$$\frac{(A-B)^2}{AB} = \frac{q}{r} \frac{[\prod_{j=1}^{J} M_j - Nr/q + o(q)/q - o(r)/q]^2}{[\prod_{j=1}^{J} M_j + o(q)/q][N + o(r)/r]}, \tag{27}$$

which is finite for all $0 \le p_i \le 1$ $i \in \Omega$ if and only if

$$q = r. \tag{28}$$

This condition is met for $P_1 = \{q_j,\ j = 1,\ldots,J\}$. For the special case $p_i = p$ $i \in \Omega$ only $|P_1| = J$ is needed.

We now turn to $\gamma(1-\bar{g}_K;\ A,B)$ in (7c) for which one has the alternative representations

$$B = 1 - \bar{q} \prod_{j=1}^{I} \bar{M}_j + o(\bar{q})$$

where

$$\bar{q} = \prod_{j=1}^{I} \bar{q}_j$$

$$\bar{q}_j = \max_{i \in P_j} (1-p_i) \qquad j = 1,\ldots,I$$

$$\bar{M}_j = \text{number of } p_i = 1 - \bar{q}_j \text{ on path } P_j \qquad j = 1,\ldots,I$$

and

$$A = 1 - \bar{N}\,\bar{r} + o(\bar{r})$$

where

$$\bar{r} = \max_{1 \le j \le J} \prod_{i \in C_j} (1-p_i)$$

and

$$\bar{N} = \text{number of cutsets } k \text{ for which } \prod_{i \in C_k} (1-p_i) = \bar{r} \quad k = 1,\ldots,J.$$

Then for a given network G

$$\frac{(A-B)^2}{(1-A)(1-B)} \approx \frac{\bar{q}}{\bar{r}} \frac{[\prod_{j=1}^{I} \bar{M}_j - \bar{N}\,\bar{r}/\bar{q} + o(\bar{r})/\bar{q} - o(\bar{q})/\bar{q}]^2}{[\prod_{j=1}^{I} \bar{M}_j + o(\bar{q})/\bar{q}][\bar{N} + o(\bar{r})/\bar{r}]}, \qquad (29)$$

which is finite for all p_i $i \in \Omega$ if and only if

$$\bar{q} = \bar{r}. \qquad (30)$$

This condition is met for $C_1 = \{1-\bar{q}_j;\ j=1,\ldots,I\}$. For the case $p_i = p$ $i \in \Omega$ only $|C_1| = I$ is needed.

These results carry considerable practical importance for they reveal conditions under which the time to achieve an estimate of g or 1-g with specified relative error is finite for a

given network G regardless of the arc probabilities $\{p_i, i\varepsilon E\}$. Moreover, the forms (27) and (29) provide valuable guidance when choosing paths and cutsets. For example, choosing paths and cutsets to mimimize $\left|\prod_{j=1}^{J} \bar{M}_j - \bar{N}\right|$ is a desirable objective with regard to (29), as is choosing them to mimimize $\left|\prod_{j=1}^{J} M_j - N\right|$ with regard to (27).

4.2 Boundedness with Respect To Network Size

When one turns to the effect of network size on the complexity of a Monte Carlo experiment with an (ε,δ) accuracy criterion, one quickly realizes that the way in which the network grows is of crucial importance. In particular, the effect on $g(s,t)$ is a central consideration. Since this topic deserves considerably more space than we can afford here, we limit our comments to a special but interesting case.

By network growth we mean an increase in the number of arcs. Recall that M and C denote the s-t path and s-t cutset of E of mimimal cardinalities. If the sizes of M and C remain constant as E grows, then it is always possible to choose a set of disjoint paths $P_1,\ldots,P_I$ $I \geq 1$ and a set of disjoint cutsets $C_1,\ldots,C_J$ $J \geq 1$ such that the resulting bounds B and A are functions only of the arc probabilities $\{p_i, i\varepsilon \sum_{j=1}^{I} P_j\}$ and $\{p_i, i\varepsilon \sum_{j=1}^{J} C_j\}$,

respectively, so that the analysis in the previous section applies. Then a Monte Carlo experiment using Procedure Q and a depth-first search algorithm to determine connectedness meets the absolute error criterion (7a) and the relative error criteria (7b) and (7c) in $O(\max(|V|, |E|))$ time.

A question remains as to how interesting is the set of networks that satisfy these restrictions on M and C. If the vertex degrees of G are bounded from above then the constraint on C is not unreasonable. However, the restriction on M may be more difficult to justify. Clearly, more remains to be said about bounds on bounds as E grows.

5. Example

An analysis of the network in Fig. 1 illustrates the benefits and costs of the VSF and KTI bounds for s-t connectedness. The network has 30 arcs, limiting the feasibility of directly calculating g.

Insert Fig. 1 about here.

The example assumes independent failures with $p_i = p$ $i \in E$, p = .5, .9 and .95, $\{s\} = 1$ and $\{t\} = 20$. For the VSF bounds we have $L = |M| = 5$ and $H = |C| = 3$. For the KTI bounds we consider two cases. Case 1 has

$$\Omega = \{1,2,3,4,9,11,18,19,27,28,29,30\}$$

$$P_1 = \{1,4,11,19,28\} \quad P_2 = \{3,9,18,27,28\}$$

$$C_1 = \{1,2,3\} \quad C_2 = \{28,29,30\}$$

and case 2 has

$E-\Omega = \{10,13,16,20,23,26\}$

$P_1 = \{3,9,18,27,28\}$ $P_2 = \{1,5,12,21,29\}$

$P_3 = \{2,7,15,24,30\}$ $C_1 = \{1,2,3\}$

$C_2 = \{28,29,30\}$ $C_3 = \{11,12,14,15,17,18\}$

$C_4 = \{4,5,6,7,8,9\}$ $C_5 = \{19,21,22,24,25,27\}$.

Observe that for case 1 $|\Omega| = 12$ enabling us to use the tabling method for $\{Q_1(y)\}$. Although case 2 with $|\Omega| = 24$ limits the possibility of tabling, P_1, P_2 and P_3 are disjoint and C_1, C_2, C_3, C_4 and C_5 are disjoint, enabling one to use Procedure Q.

Table 1 lists the bounds $1-A$ and $1-B$ together with the

Insert Table 1 about here.

bounds on sample size that (10) induces for $\delta = .05$ and $\varepsilon = .05$. Observe that these worst case VSF bounds are of little value compared to the KTI bounds. Also, note that the advantage of case 2 relative to case 1, with regard to a bound on K, increases with p.

One measure of the effect of a variance reducing technique is the variance ratio $g(1-g)/K \text{ var } \bar{g}_K$. A ratio greater than unity indicates that the technique has the desired effect. A time ratio T_1/T_2 is also used where T_1 denotes the mean time to collect an observation using a crude Monte Carlo sampling plan and T_2 denotes the mean time to collect an observation using the proposed method. Then $[g(1-g)/K \text{ var } \bar{g}_K] \times T_1/T_2$ gives the relative time required to achieve a given variance with crude Monte Carlo sampling as compared to the time required to achieve this same variance with the proposed method.

Table 2 shows these ratios for the VSF and KTI bounds and $p = .5, .9, .95$ for $K = 2^{16} = 262144$ independent replications using Procedure BIN for the VSF case, a table of $\{Q_1(y)\}$ together with Procedure BIN for KTI case 1 and Procedure Q for KTI case 2.

Insert Table 2 about here.

The results indicate that both the VSF and KTI bounds have advantages, but the KTI bounds clearly dominate. Moreover, observe that Procedure Q consumes considerably more time than tabling does. Also note the a priori minimal variance ratio $g'(1-g')/(A-g')(g'\neg B)$ in Table 2.

6. Assessing Credibility

Since the worst case bounds of Section 2 often lead to considerably larger sample sizes K than are required in specific analyses, one is interested in ways of assessing the extent to which a specific criterion is being met as the sampling experiment progresses. Although a statistical literature does exist for sequential sampling to achieve a fixed width confidence interval, (e.g. Chow and Robbins 1965) the fact that it relies on asymptotic behavior ($\varepsilon \to 0$) limits its appeal. As an alternative we describe how one can assess the credibility of an absolute or relative error criterion as the experiment proceeds.

Recall that $g = (A\neg B)\mu + B$. Therefore, criteria (1),(2) and (3) imply $\mu \in [\mu_*, \mu^*]$ where for (1)

$$\mu_* = \max(0, \frac{S}{K} - \frac{\varepsilon}{A-B})$$

$$\mu^* = \min(1, \frac{S}{K} + \frac{\varepsilon}{A-B}),$$

for (2)

$$\mu_* = \max[0, \frac{1}{1-\varepsilon}(\frac{S}{K} - \frac{\varepsilon B}{A-B})]$$

$$\mu^* = \min[1, \frac{1}{1+\varepsilon}(\frac{S}{K} + \frac{\varepsilon B}{A-B})]$$

and for (3)

$$\mu_* = \max\{0, \frac{1}{1-\varepsilon}[\frac{S}{K} - \frac{\varepsilon(1-B)}{A-B}]\}$$

$$\mu^* = \min\{1, \frac{1}{1+\varepsilon}[\frac{S}{K} + \frac{\varepsilon(1-B)}{A-B}]\}.$$

Since S has the binomial distribution one has

$$pr\{\mu \in [\mu_*, \mu^*]\} = 1 - F_S(K, \mu_*) + F_S(K, \mu^*), \qquad (31)$$

F being the binomial distribution function.

Table 3 shows this probability for a relative error criterion $\varepsilon = .05$ for $1-g$ for case 2 of the KTI bounds. By choosing powers of 2 for K we insure that at each successive evaluation half the data provides new information. Observe that one may be

Insert Table 3 about here.

confident in terminating the sampling experiment at K = 1024 for p = .5 and at K = 32768 for p = .9 and .95. IMSL (1982) and SAS (1982) provide routines for evaluating the binomial distribution.

7. Confidence Intervals

While the credibility analysis of Section 6 provides considerable guidance, one would also like to evaluate the precision of $\bar{g}_K$ more explicitly at the end of an experiment. Confidence intervals provide one way of making this assessment. As the next two subsections show, different methods exist for constructing these intervals. Section 7.1 describes an exact method that requires some care with regard to computation. Section 7.2 describes a relatively simple method whose results rely on either general bounds or asymptotic limits.

7.1 Exact Confidence Intervals

Suppose that exactly S successes occur in K independent trials. Then there exist 2-tuples $(\mu_1, \alpha_1,)$ and (μ_2, α_2) with $0 < \mu_1 < \mu_2 < 1$ and $0 < \alpha_1 < \alpha_2 < 1$ such that

$$1 - F_{S-1}(K,\mu_1) = \alpha_1$$

and

$$F_S(K,\mu_2) = 1 - \alpha_2.$$

Let

$$\Theta(\alpha_1) = (\mu_1 : F_{S-1}(K,\mu_1) = 1 - \alpha_1)$$

and (32)

$$\Theta(\alpha_2) = (\mu_2 : F_S(K,\mu_2) = 1 - \alpha_2).$$

Then $[\theta(\alpha_1), \theta(\alpha_2)]$ is a confidence interval for μ at confidence level $\alpha = \alpha_2 - \alpha_1$. Since there are many $\alpha_1 < \alpha_2$ that satisfy $\alpha = \alpha_2 - \alpha_1$, one way to proceed is to choose α_1 and α_2 to minimize $\theta(\alpha_2) - \theta(\alpha_1)$, subject to $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha = \alpha_2 - \alpha_1$, thus giving the shortest interval at level α. An alternative way to proceed is to choose $\alpha_1 = (1-\alpha)/2$ and $\alpha_2 = (1+\alpha)/2$ so that each tail has equal probability.

To facilitate the determination of $\theta(\alpha_1)$ and $\theta(\alpha_2)$ in (32) observe that (Abramowitz and Stegun 1964, p. 945)

$$1 - F_{j-1}(n,\theta) = \int_0^{\theta} \binom{n}{j} w^{j-1}(1-w)^{n-j} dw. \qquad (33)$$

Therefore $\theta(\alpha_1)$ calls for an evaluation of the inverse Beta distribution with parameters S and K = S + 1 and $\theta(\alpha_2)$, for an evaluation of the inverse Beta distribution with parameters S + 1 and K = S. Procedure INTERVAL describes how to compute the shortest confidence interval at level α. Although standard computing packages exist to perform the inversions for $\theta(\alpha_i)$ i = 1,2 in step IIc, experience with such packages in IMSL(1982) and SAS(1982), indicates that a large K severely limits their abilities to produce numerical results when S<<K. It is precisely in such cases that one may be willing to settle for bounding or approximating results.

Procedure INTERVAL

Purpose: To compute a confidence interval for g at level α.

Input: Sample size K, number of successes S, bounds A and B, confidence level α and grid size N.

Output: α confidence interval $[g_*, g^*]$ of shortest length.

Method:

I. Initialization

a. $L = 0$ and $U = 1$.

II. Search for shortest interval

For each $i = 1, \ldots, N-1$ do:

a. $\alpha_1 = (1-\alpha)(i/N)$.

b. $\alpha_2 = \alpha + \alpha_1$.

c. For $j = 1,2$ solve (31) for $\theta(\alpha_j)$.

d. If $U - L > \theta(\alpha_2) - \theta(\alpha_1)$, then set $L = \theta(\alpha_1)$ and $U = \theta(\alpha_1)$.

III. Compute confidence interval

a. $g_* = (A-B)L + B$.

b. $g^* = (A-B)U + B$.

End of procedure.

7.2 Intervals Based on Probability Inequalities and Limits

As Section 1 indicates, inequalities and approximations exist that can be used to compute worst case upper bounds for K for specified δ and ε. These inequalities also are of value when computing confidence intervals. Consider the probability statement

$$\mathrm{pr}\left\{ \frac{|\bar{g}_K - g|}{[(A-g)(g-B)/K]^{1/2}} \leq \beta(1-\alpha) \right\} \geq \alpha \quad 0 < \alpha < 1. \tag{34}$$

By rearranging terms in the argument of $\mathrm{pr}\{\cdot\}$ in (34), one can assert that with probability of at least α the interval

$$\frac{(A-B)S + KB + (A+B)\beta^2/2 \pm \beta(A-B)[\beta^2/4 + S(K-S)/K]^{1/2}}{K + \beta^2} \tag{35}$$

covers g, where $\beta \equiv \beta(1-\alpha)$. As in the case of worst case bounds one determines $\beta(1-\alpha)$ for a specified confidence level α from (11) for Chebyshev's inequality and from (12) for the normal approximation. Table 4 presents the bounding and approximating .95 confidence intervals for 1-g for the network in Fig. 1.

Insert Table 4 about here.

8. Essential Steps for Implementation

The results of this paper show that an effective Monte Carlo procedure, based on bounds B and A, exists for estimating g(s,T) in worst case time $O(\max(|V|, |E|) \cdot D)$ time where $D = (A-B)^2$ for a given (ε,δ) absolute error criterion and $D = (A-B)^2/\min[AB,(1-A)(1-B)]$ for a given (ε,δ) relative error criterion. Moreover, the number of replications required to achieve these worst case bounds can be computed prior to experimentation and used as a guide. Of the two methods of deriving bounds described here, the example of Section 4 shows that the KTI bounds are most beneficial, especially for p close to unity. Therefore, we recommend that the KTI bounds be used in practice together with the following steps:

1. Determine a set of edge-disjoint s-t paths $P_1,\ldots,P_I$.
2. Determine a set of edge-disjoint s-t cutsets $C_1,\ldots,C_J$.
3. Compute B from $P_1,\ldots,P_I$ in $O(|\sum_{j=1}^{I} P_j|)$ time and A from $C_1,\ldots,C_J$ in $O(|\sum_{j=1}^{J} C_j|)$ time.
4. Choose a (ε,δ) absolute or relative error criterion ((1),(2) or (3)) and assign values to ε and δ. Then determine the bound K^* on K accordingly from (8), (9) or (10). Use this value for guidance.
5. If $|\Omega| = |(\sum_{j=1}^{I} P_j) \cup (\sum_{j=1}^{J} C_j)|$ is small enough, then compute a table of the probability mass function $\{Q_1(y)\}$ in

$O(2^{|\Omega|})$ time.

6. For the sampling experiment use Procedure BOUNDS.

   a. Table approach:

      Sample arcs in $E-\Omega$ using Procedure B if the p_i are distinct and using Procedure BIN if they are equal. Sample arcs in Ω from a table of $\{Q_1(y)\}$.

   b. Sequential approach: Use Procedure Q.

      Steps a and b each take $O(|E|)$ time.

   c. Check for connectedness using a depth-first search in $O(\max(|V|, |E|))$ time.

7. If the sampling experiment is to be performed in blocks of K_1, $K_1 + K_2$, $K_1 + K_2 + K_3$, replications, etc., then after each block compute the credibility probability (31). Suggested increments are $K_i = K_1 2^{i-1}$ for $i = 1,2,\ldots$

8. After completion of the experiment compute a confidence interval for g, as in Section 6.

Acknowledgements

I would like to express my gratitude to E. Scott Provan for our many helpful discussions and to Tien Yi Shaw for his able programming assistance.

References

1. Abramowitz, M. and I. Stegun (1964). Handbook of Mathematical Functions, Applied Mathematics Series 55, National Bureau of Standards.

2. Aho, A.V., J.E. Hopcroft and J.D. Ullman (1974). The and Analysis of Computer Algorithms, Addison-Wesley.

3. Chow, Y.S. and H. Robbins (1965). "On the Asympotiti Theory of Fixed Width Sequential Confidence Intervals the Mean," Ann. Math. Statist., 36, 457-462.

4. Easton, M.C. and C.K Wong (1980). "Sequential Destru Method for Monte Carlo Evaluation of System Reliabili IEEE Transactions on Reliability, R-29, 27-32.

5. Esary, J.D. and F. Proschan (1966). "Coherent Struct Non-Identical Components," Technometrics, 5, 191-209.

6. Fishman, G.S. (1983a). "An Alternative Perspective o Carlo Estimation of Network Reliability", Technical R UNC/ORSA/TR-83/2, Curriculum in Operations Research a Systems Analysis, University of North Carolina, Chape

7. Fishman, G.S. (1983b). "Estimating Network Reliabili Accelerated Convergence Rates on Error Bounds." Techn Report UNC/ORSA/TR-83/7, Curriculum in Operations Res and Systems Analysis, University of North Carolina, C Hill.

8. Fishman, G.S. and L. R. Moore (1984). "Sampling from Discrete Distribution While Preserving Monotonicity," American Statistician, to appear.

9. IMSL, Inc. (1982). IMSL Library Reference Manual, Ed Houston Texas,

10. Karp, R. and M.G. Luby (1983). "A New Monte Carlo Me for Estimating the Failure Probability of an n-Compon System", Computer Science Division, University of California, Berkeley.

11. Katona, G. (1966). "A Theorem of Finite Sets", Theor Graphs, Proc. Tihany Colloquium, September 1966, P. E and G. Katona, eds., 187-207.

12. Kronmal, R.A. and A.V. Peterson (1979). "On the Alia Method for Generating Random Variables from a Discret Distribution," The American Statistician 33, 214-218.

13. Kumamoto, H., K. Tanaka and K. Inoue (1977). "Efficient Evaluation of System Reliability by Monte Carlo Method," IEEE Transactions on Reliability, R-26, 311-315.

14. Kumamoto, H., K. Tanaka, K. Inoue and E.J. Henley (1980). "Dagger-Sampling Monte Carlo for System Unavailability Evaluation", IEEE Transactions on Reliability, R-29, 122-125.

15. Kruskal, J.G. (1963). "The Number of Simplices in a Complex", Mathematical Optimization Techniques, R. Bellman, ed., University of California, Berkeley, 251-178.

16. Papadimitriou, C.H. and K. Steiglitz (1982). Combinatorial Optimization, Prentice-Hall.

17. SAS Institute, Inc. (1982). SAS User's Guide: Basics, Cary, North Carolina.

18. Van Slyke, R. and H. Frank (1972). "Network Reliability Analysis," Networks, 1, 279-290.

19. Wagner, H.M. (1975). Principles of Operations Research, second ed., Prentice-Hall.

20. Walker, A.J. (1977). "An Efficient Method for Generating Discrete Random Variables with General Distributions," ACM Transactions on Mathematical Software, 3, 253-256.

Table 1
Bounds for Relative Accuracy Criterion (10)
$\delta = .05$ and $\epsilon = .05$

	1-B	1-A	Bounds on K Chebyshev	Normal
p=.50				
VSF	$.1000 \times 10^{1}$	$.2974 \times 10^{-4}$	67×10^{6}	13×10^{6}
KTI 1	.9395	.2344	4515	867
2	.9092	.2697	3335	640
p=.90				
VSF	.5886	$.1828 \times 10^{-21}$	64×10^{23}	12×10^{23}
KTI 1	.2064	$.1999 \times 10^{-2}$	202522	38898
2	$.6867 \times 10^{-1}$	$.2002 \times 10^{-2}$	64659	12419
p=.95				
VSF	.1878	$.3352 \times 10^{-29}$	11×10^{31}	21×10^{30}
KTI 1	$.8269 \times 10^{-1}$	$.3031 \times 10^{-3}$	541635	101671
2	$.1158 \times 10^{-1}$	$.2500 \times 10^{-3}$	88683	17033

Table 2
Variance Reductions †
K = 262144

	$1-\bar{g}_K$	$\frac{g(1-g)}{K\mathrm{var}\bar{g}_K}$	$\frac{g'(1-g')}{(A-g')(g'-B)}$	$\frac{T_1}{T_2}$	$\frac{g(1-g)}{K\mathrm{var}\bar{g}_K} \times \frac{T_1}{T_2}$
p=.50					
VSF	.7104	1.00	1.00	.94	.94
KTI 1	.7094	1.89	1.88	1.04	1.97
KTI 2	.7105	2.35	2.31	.24	.56
p=.90					
VSF	$.2933\times10^{-2}$	1.70	1.70	.91	1.55
KTI 1	$.2862\times10^{-2}$	16.67	5.83	1.00	16.67
KTI 2	$.2867\times10^{-2}$	50.22	20.93	.26	13.06
p=.95					
VSF	$.2938\times10^{-3}$	5.33	5.32	.82	4.37
KTI 1	$.2887\times10^{-3}$	95.04	13.63	.97	92.19
KTI 2	$.2940\times10^{-3}$	592.23	118.46	.23	135.49

† g is estimated by $\bar{g}_K$ in the quantity g(1-g) and var $\bar{g}_K$ is estimated by $V(\bar{g}_K)$.

Table 3
Credibility Results

$pr[|\bar{g}_K - g| > .05(1-g)]$

KTI Bounds, Case 2

K	p=.5	p=.9	p=.95
2	.869	1.000	1.000
4	.755	1.000	1.000
8	.615	1.000	1.000
16	.671	1.000	1.000
32	.504	.927	1.000
64	.343	.892	1.000
128	.197	.841	1.000
256	$.657\times10^{-1}$	.778	1.000
572	$.821\times10^{-2}$	.664	.531
1024	$.242\times10^{-3}$	.561	.559
2045	$.281\times10^{-6}$	.398	.407
4096	$.371\times10^{-12}$	.223	.219
8192	$.217\times10^{-23}$	$.802\times10^{-1}$	$.848\times10^{-1}$
16384	$.235\times10^{-45}$	$.134\times10^{-1}$	$.126\times10^{-1}$
32768	.000	$.559\times10^{-3}$	$.248\times10^{-3}$
65536	.000	$.858\times10^{-6}$	$.123\times10^{-6}$
131072	.000	$.160\times10^{-11}$	$.196\times10^{-15}$
262144	.000	$.336\times10^{-22}$	$.114\times10^{-29}$

Table 4
.95 Confidence Intervals for 1-g

	Chebyshev	Normal
KTI 1		
p=.50		
lower	.7065	.7087
upper	.7123	.7107
width	$.5775 \times 10^{-2}$	$.2531 \times 10^{-2}$
p=.90		
lower	$.2754 \times 10^{-2}$	$.2813 \times 10^{-2}$
upper	$.2986 \times 10^{-2}$	$.2914 \times 10^{-2}$
width	$.2321 \times 10^{-3}$	$.1015 \times 10^{-3}$
p=.95		
lower	$.2759 \times 10^{-3}$	$.2824 \times 10^{-3}$
upper	$.3077 \times 10^{-3}$	$.2961 \times 10^{-3}$
width	$.3181 \times 10^{-4}$	$.1372 \times 10^{-4}$
KTI 2		
p=.50		
lower	.7079	.7094
upper	.7131	.7117
width	$.5169 \times 10^{-2}$	$.2265 \times 10^{-2}$
p=.90		
lower	$.2804 \times 10^{-2}$	$.2839 \times 10^{-2}$
upper	$.2935 \times 10^{-2}$	$.2896 \times 10^{-2}$
width	$.1319 \times 10^{-3}$	$.6054 \times 10^{-4}$
p=.95		
lower	$.2883 \times 10^{-3}$	$.2914 \times 10^{-3}$
upper	$.3066 \times 10^{-3}$	$.2968 \times 10^{-3}$
width	$.1234 \times 10^{-4}$	$.5396 \times 10^{-5}$

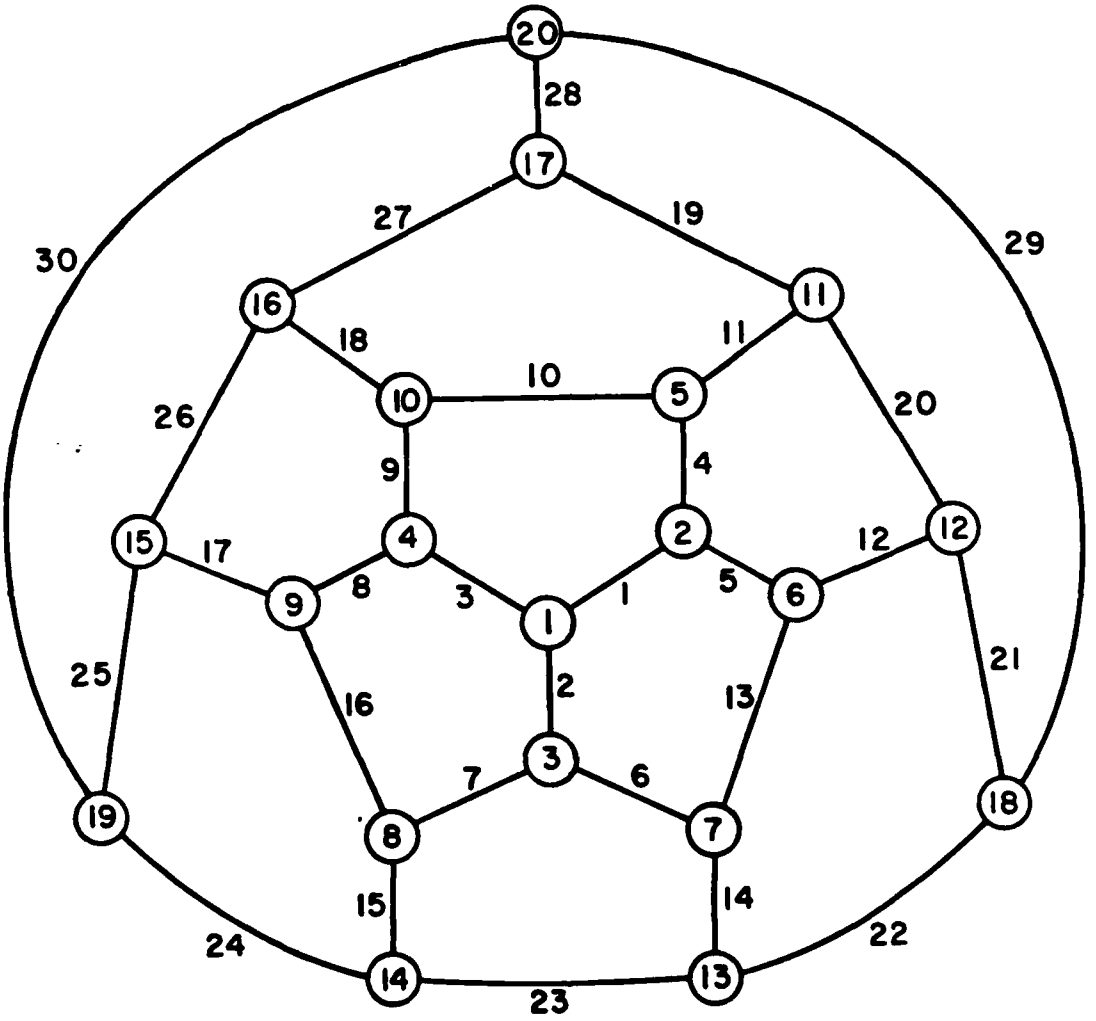

Fig.1 Network

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) UNC/ORSA/TR-84-8	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 85-0046
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State and ZIP Code) Curriculum in Operations Research and Systems Analysis, Smith Building 128A, Chapel Hill NC 27514	7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-84-0140
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NOS. PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT NO.
11. TITLE (Include Security Classification) A MONTE CARLO SAMPLING PLAN FOR ESTIMATING NETWORK RELIABILITY	
12. PERSONAL AUTHOR(S) George S. Fishman	
13a. TYPE OF REPORT Technical 13b. TIME COVERED FROM _____ TO _____	14. DATE OF REPORT (Yr., Mo., Day) OCT 84 15. PAGE COUNT 45
16. SUPPLEMENTARY NOTATION	
17. COSATI CODES FIELD / GROUP / SUB. GR.	18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) Monte Carlo methods; network reliability; variance reduction.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Consider an acyclic undirected network G = (V,E) with node set V and arc set E whose arcs are subject to random failure. Let s be a node in V and T a set of nodes in V such that s ∉ T. This paper presents a relatively complete and comprehensive description of a ganeral class of Monte Carlo sampling plans for estimating g = g(s,T), the probability that s is connected to all nodes in T. The paper also provides procedures for implementing these plans. Each plan uses known lower and upper bounds [B,A] on g to produce an estimator of g that has a smaller variance (A-g)(g-B)/K than one obtains for crude Monte Carlo sampling (B=0, A=1) on K independent replications. The paper describes worst case bounds on sample sizes K, in terms of B and A, for meeting absolute and relative error (CONTINUED)

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22a. NAME OF RESPONSIBLE INDIVIDUAL MAJ Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code) (202) 767- 5027 22c. OFFICE SYMBOL NM

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ITEM #19, ABSTRACT, CONTINUED: criteria. It also gives the worst case bound on the amount of variance reduction that can be expected when compared with crude Monte Carlo sampling.

Two plans are studied in detail for the case T = {t}. An example illustrates the variance reductions achievable with these plans. The paper next shows how to assess the credibility that a specified error criterion for g is met as the Monte Carlo experiment progresses and then shows how confidence intervals can be computed for g. Lastly, the paper summarizes the steps needed to implement the proposed techniques.

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